

On zero neighbours and trial sets of linear codes

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Abstract—In this work we study the set of leader codewords of a non-binary linear code. This set has some nice properties related to the monotonicity of the weight compatible order on the generalized support of a vector in \mathbb{F}_q^n . This allows us to describe a test set, a trial set and the set zero neighbours in terms of the leader codewords.

Index Terms—Linear codes, monotone functions, test set, trial set, zero neighbours

I. INTRODUCTION

AS it is pointed in [4] it is common folklore in the theory of binary codes that there is an ordering on the coset leaders chosen as the lexicographically smallest minimum weight vectors that provides a monotone structure. In the binary case this is expressed as follows: if \mathbf{x} is a coset leader and $\mathbf{y} \subseteq \mathbf{x}$ (i.e. $y_i \leq x_i$ for all i) then \mathbf{y} is also a coset leader. This nice property has been proved of great value, see for example [6], and it has been used for analyzing the error-correction capability of binary linear codes [4]. In this last paper the authors introduce the concept of a *trial set* of codewords and they provide a gradient-like decoding algorithm based on this set.

Despite the interest of this topic no generalization of this ideas is known by the authors of this communication to the q -ary linear case, that is, to codes over \mathbb{F}_q the field of $q = p^r$ elements where p is a prime. In this paper we provide a non straightforward generalization of this ideas to the q -ary case. In Section II we introduce the idea of a generalized support of a vector in \mathbb{F}_q^n based on considering \mathbb{F}_q as a \mathbb{F}_p vector space using the so called p -ary expansion. Section III is devoted to the study of the leader codewords of a code as a zero neighbor set and their properties. The leader codewords have been previously defined for the binary case in [3] but the analogous concept in the q -ary case needs a subtle and non-trivial generalization as well as their properties. Finally in Section IV we analyse the correctable and uncorrectable errors defining a trial set for a linear code from the set of leader codewords.

II. PRELIMINARIES

From now on we shall denote by \mathbb{F}_q the finite field with q elements, with $q = p^m$ and p a prime. A *linear code* \mathcal{C} over \mathbb{F}_q of length n and dimension k , or an $[n, k]$ linear code for short,

is a k -dimensional subspace of \mathbb{F}_q^n . We will call the vectors \mathbf{v} in \mathbb{F}_q^n words and the particular case where $\mathbf{v} \in \mathcal{C}$, codewords. For every word $\mathbf{v} \in \mathbb{F}_q^n$ its *support* is define as its support as a vector in \mathbb{F}_q^n , i.e. $\text{supp}(\mathbf{v}) = \{i \mid v_i \neq 0\}$ and its *Hamming weight*, denoted by $w_H(\mathbf{v})$ as the cardinality of $\text{supp}(\mathbf{v})$.

As usual the *Hamming distance*, $d_H(\mathbf{x}, \mathbf{y})$, between two words $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ is the number of places where they differ, or equivalently, $d_H(\mathbf{x}, \mathbf{y}) = w_H(\mathbf{x} - \mathbf{y})$. The *minimum distance* $d(\mathcal{C})$ of a linear code \mathcal{C} is defined as the minimum weight among all nonzero codewords.

The words of minimal Hamming weight in the cosets of $\mathbb{F}_q^n/\mathcal{C}$ is the *set of coset leaders* of the code \mathcal{C} in \mathbb{F}_q^n and we will denote it by $\text{CL}(\mathcal{C})$. $\text{CL}(\mathbf{y})$ will denote the subset of coset leaders corresponding to the coset $\mathbf{y} + \mathcal{C}$. Given a coset $\mathbf{y} + \mathcal{C}$ we define the *weight of the coset* $w_H(\mathbf{y} + \mathcal{C})$ as the smallest Hamming weight among all vectors in the coset, or equivalently the weight of one of its leaders. It is well known that if $t = \lfloor \frac{d(\mathcal{C})-1}{2} \rfloor$ is the *error-correcting capacity* of \mathcal{C} where $\lfloor \cdot \rfloor$ denotes the greatest integer function then every coset of weight at most t has a unique coset leader.

We will assume that $\mathbb{F}_q = \frac{\mathbb{F}_p[X]}{(f(X))}$ where $f(X)$ is chosen such that $f(X)$ is an irreducible polynomial over \mathbb{F}_p of degree m . Let β be a root of $f(X)$, then an equivalent representation of \mathbb{F}_q is $\mathbb{F}_p[\beta]$, i.e. any element of $a \in \mathbb{F}_q$ is represented as

$$a_1 + a_2\beta + \dots + a_m\beta^{m-1} \text{ with } a_i \in \mathbb{F}_p \text{ for } i \in \{1, \dots, m\}.$$

For a word $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{F}_q^n$, such that the i -th component of \mathbf{v} is

$$v_i = a_{i,1} + a_{i,2}\beta + \dots + a_{i,m}\beta^{m-1}$$

we define the *generalized support* of a vector \mathbf{v} as the support of the nm -tuple given by the concatenations of the p -adic expansion of each component \mathbf{v}_i of \mathbf{v} , i.e.

$$\text{supp}_{\text{gen}}(\mathbf{v}) = \{\text{supp}((a_{i,0}, \dots, a_{i,m-1})) : i = 1 \dots n\},$$

$$\text{and } \text{supp}_{\text{gen}}(\mathbf{v})[i] = \text{supp}((a_{i,0}, \dots, a_{i,m-1})).$$

We will say that $(i, j) \in \text{supp}_{\text{gen}}(\mathbf{v})$ if the corresponding $a_{i,j}$ is not zero. From now on the set $\{\mathbf{e}_{ij} = \beta^{j-1}\mathbf{e}_i : i = 1, \dots, n; j = 1, \dots, m\}$ will be denoted as $\text{Can}(\mathbb{F}_q, f)$ and it represents the canonical basis of $(\mathbb{F}_q^n, +)$, the additive monoid \mathbb{F}_q^n with respect to the “+” operation, where f is the irreducible polynomial used to define \mathbb{F}_q^n .

We will say that a representation of a word \mathbf{w} as an nm -tuple is in *standard form* if it can not be reduced with respect to the “+” operation in the representation, i.e., it is the minimal representation for \mathbf{w} with respect to the generalized support. We will denote the standard form of \mathbf{v} as $\text{SF}(\mathbf{v}, f)$. Therefore, \mathbf{w} is in standard form if $\mathbf{w} \equiv \text{SF}(\mathbf{w}, f)$ (we will also say $\mathbf{w} \in \text{SF}(\mathbb{F}_q^n, f)$), note that we do not use common equality

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symbol in order to emphasize that the equality is considered as literal expressions in terms of the basis $\text{Can}(\mathbb{F}_q, f)$.

Remark 1. From now on we will use $\text{Can}(\mathbb{F}_q)$ and $\text{SF}(\mathbb{F}_q^n)$ instead of $\text{Can}(\mathbb{F}_q, f)$ and $\text{SF}(\mathbb{F}_q^n, f)$ respectively since it is clear that different elections of f or β provide equivalent generalized supports.

Given $\mathbf{x}, \mathbf{y} \in (\mathbb{F}_q^n, +)$, $\mathbf{x} = \sum_{i,j} x_{ij} \mathbf{e}_{ij}$, $\mathbf{y} = \sum_{i,j} y_{ij} \mathbf{e}_{ij}$, $\mathbf{x} \subset \mathbf{y}$ if $x_{ij} \leq y_{ij}$ for all $i \in [1, n]$, and $j \in [1, m]$. An *admissible order* on $(\mathbb{F}_q^n, +)$ is a total order $<$ on $(\mathbb{F}_q^n, +)$ satisfying the following two conditions

- 1) $\mathbf{0} < \mathbf{x}$, for all $\mathbf{x} \in (\mathbb{F}_q^n, +)$, $\mathbf{x} \neq \mathbf{0}$.
- 2) If $\mathbf{x} < \mathbf{y}$, then $\mathbf{x} \oplus \mathbf{z} < \mathbf{y} \oplus \mathbf{z}$, for all $\mathbf{z} \in (\mathbb{F}_q^n, +)$, where \oplus denotes the operation of grouping the same canonical words of $\text{Can}(\mathbb{F}_q)$ but without reducing the coefficients module p .

Definition 1. We define the *Voronoi region* of a codeword $\mathbf{c} \in \mathcal{C}$ and we denote it by $D(\mathbf{c})$ as the set

$$D(\mathbf{c}) = \{\mathbf{y} \in \mathbb{F}_q^n \mid d_H(\mathbf{y}, \mathbf{c}) \leq d_H(\mathbf{y}, \mathbf{c}'), \forall \mathbf{c}' \in \mathcal{C} \setminus \{\mathbf{0}\}\}.$$

Note that the set of all the Voronoi regions for a given linear code \mathcal{C} covers the space \mathbb{F}_q^n and also it is clear that $D(\mathbf{0}) = \text{CL}(\mathcal{C})$. However, some words in \mathbb{F}_q^n may be contained in several regions.

For any subset $A \subset \mathbb{F}_q^n$ we define $\mathcal{X}(A)$ as the set of words at Hamming distance 1 from A , i.e.

$$\mathcal{X}(A) = \{\mathbf{y} \in \mathbb{F}_q^n \mid \min \{d_H(\mathbf{y}, \mathbf{a}) : \mathbf{a} \in A\} = 1\}.$$

We define the *boundary* of A as $\delta(A) = \mathcal{X}(A) \cup \mathcal{X}(\mathbb{F}_2^n \setminus A)$.

Definition 2. A nonzero codeword $\mathbf{c} \in \mathcal{C}$ is called a *zero neighbour* if its Voronoi region shares a common boundary with the set of coset leaders, i.e.

$$\delta(D(\mathbf{z})) \cap \delta(D(\mathbf{0})) \neq \emptyset.$$

We will denote by $\mathcal{Z}(\mathcal{C})$ the set of all zero neighbours of \mathcal{C}

$$\mathcal{Z}(\mathcal{C}) = \{\mathbf{z} \in \mathcal{C} \setminus \{\mathbf{0}\} : \delta(D(\mathbf{z})) \cap \delta(D(\mathbf{0})) \neq \emptyset\}.$$

Definition 3. A *test-set* \mathcal{T} for a given linear code \mathcal{C} is a set of codewords such that every word \mathbf{y}

- 1) either \mathbf{y} lies in $D(\mathbf{0})$, the Voronoi region of the all-zero vector,
- 2) or there exists $\mathbf{v} \in \mathcal{T}$ such that $w_H(\mathbf{y} - \mathbf{v}) < w_H(\mathbf{y})$.

The set of zero neighbours is a test set, also from the set of zero neighbours can be obtained any minimal test set according to the cardinality of the set [1].

III. ZERO NEIGHBOURS AND LEADER CODEWORDS

The first idea that allow us to compute incrementally de set of all coset leaders for a linear code was introduced in [2]. In that paper we used the additive structure of \mathbb{F}_q^n with the set of canonical generators $\text{Can}(\mathbb{F}_q)$. Unfortunately in [2] most of the chosen coset representatives may not be coset leaders if the weight of the coset is greater than the error-correcting capability of the code.

Theorem 1. Let $\mathbf{w} \in \text{SF}(\mathbb{F}_q^n)$ be an element in $\text{CL}(\mathcal{C})$, and $(i, j) \in \text{supp}_{\text{gen}}(\mathbf{w})$. Let $\mathbf{y} \in \text{SF}(\mathbb{F}_q^n)$ s.t. $\mathbf{w} = \mathbf{y} + \mathbf{e}_{ij}$ then

$$w_H(\mathbf{y}) \leq w_H(\mathbf{y} + \mathcal{C}) + 1.$$

The proof of the Theorem is analogous to the binary case in [3, Theorem 2.1]. In the situation of Theorem 1 above we will say that the coset leader \mathbf{w} is an *ancestor* of the word \mathbf{y} , and that \mathbf{y} is a *descendant* of \mathbf{w} . In the binary case this definitions behave as the ones in [5, §11.7] but in the case $q \neq 2$ there is a subtle difference, a coset leader could be an ancestor of a coset leader or an ancestor of a word at Hamming distance 1 to a coset leader (this last case is not possible in the binary case).

Thus, in order to incrementally generate all coset leaders starting from $\mathbf{0}$ the all zero codeword and adding elements in $\text{Can}(\mathbb{F}_q)$, we must consider all words with distance 1 to a coset leader. Note that to achieve all words at distance one from the coset leaders some words at distance 2 are needed.

Theorem 2.

- 1) Let \mathbf{w} be a word in \mathbb{F}_q^n such that $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) = 1$ and $\mathbf{w} = \mathbf{y} + \mathbf{e}_{ij}$ for some word $\mathbf{y} \in \mathbb{F}_q^n$, $(i, j) \in \text{supp}_{\text{gen}}(\mathbf{w})$ and $\mathbf{w}, \mathbf{y} \in \text{SF}(\mathbb{F}_q^n)$, then

$$w_H(\mathbf{y}) \leq w_H(\mathbf{y} + \mathcal{C}) + 2. \quad (1)$$

- 2) If equality holds in Eq. (1) then $\text{supp}_{\text{gen}}(\mathbf{y})[i] \neq \emptyset$ and $\text{supp}_{\text{gen}}(\mathbf{v})[i] = \emptyset$ for any $\mathbf{v} \in \text{CL}(\mathbf{y})$.

Note that in the second case, i.e. when equality in Eq. (1) holds then the condition implies that $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) = 1$. It is clear that if $w_H(\mathbf{y}) > w_H(\mathbf{y} + \mathcal{C}) + 2$ then $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) > 1$ since the addition of a \mathbf{e}_{ij} can only modify the distance from -1 to 1.

A. Weight compatible order

Given \prec_1 an admissible order on $(\mathbb{F}_q^n, +)$ we define the *weight compatible order* \prec on $(\mathbb{F}_q^n, +)$ associated to \prec_1 as the ordering given by

- 1) $\mathbf{x} \prec \mathbf{y}$ if $w_H(\mathbf{x}) < w_H(\mathbf{y})$ or
- 2) if $w_H(\mathbf{x}) = w_H(\mathbf{y})$ then $\mathbf{x} \prec_1 \mathbf{y}$.

I.e. the words are ordered according their weights and \prec_1 break ties. This class of orders is a subset of the class of α -orderings monotone in [4]. In fact we will need a little more than monotonicity, for the purpose of this work we will also need that for every pair $\mathbf{a}, \mathbf{b} \in \text{SF}(\mathbb{F}_q^n)$

$$\text{if } \mathbf{a} \subset \mathbf{b}, \text{ then } \mathbf{a} \prec \mathbf{b}. \quad (2)$$

Note that for any weight compatible ordering \prec every strictly decreasing sequence terminates (due to the finiteness of the set \mathbb{F}_q^n) and the condition in Eq. (2) is fulfilled.

Definition 4. We define the object **List** as an ordered set of elements in \mathbb{F}_q^n w.r.t. a weight compatible order \prec verifying the following properties:

- 1) $\mathbf{0} \in \text{List}$.
- 2) Criterion 1: If $\mathbf{v} \in \text{List}$ and $w_H(\mathbf{v}) = w_H(N(\mathbf{v}))$ then $\{\mathbf{v} + \mathbf{e}_{ij} \mid \mathbf{v} + \mathbf{e}_{ij} \in \text{SF}(\mathbb{F}_q^n)\} \subset \text{List}$.

- 3) Criterion 2: If $\mathbf{v} \in \text{List}$ and $w_H(\mathbf{v}) = w_H(N(\mathbf{v})) + 1$ then $\{\mathbf{v} + \mathbf{e}_{ij} \mid \mathbf{v} + \mathbf{e}_{ij} \in \text{SF}(\mathbb{F}_q^n)\} \subset \text{List}$.
- 4) Criterion 3: If $\mathbf{v} \in \text{List}$ and $w_H(\mathbf{v}) = w_H(N(\mathbf{v})) + 2$ then $\{\mathbf{v} + \mathbf{e}_{ij} \mid (i, j) \in I\} \subset \text{List}$, where $I = \{(i, j) \in \text{supp}_{\text{gen}}(\mathbf{v}) \mid \mathbf{v} + \mathbf{e}_{ij} \in \text{SF}(\mathbb{F}_q^n), \text{supp}_{\text{gen}}(\mathbf{v}')[i] = \emptyset, \text{ for all } \mathbf{v}' \in \text{CL}(\mathbf{v})\}$,

where $N(\mathbf{v}) = \min_{\prec} \{\mathbf{w} \mid \mathbf{w} \in \text{List} \cap (\mathcal{C} + \mathbf{v})\}$. We denote by \mathcal{N} the set of distinct $N(\mathbf{v})$ with $\mathbf{v} \in \text{List}$.

Remark 2. Observe that if $\mathbf{v} \in \mathbb{F}_q^n$ satisfies Criterion 1 in Definition 4 then $\mathbf{v} \in \text{CL}(\mathcal{C})$. In particular, when \mathbf{v} is the first element of List that belongs to $\mathcal{C} + \mathbf{v}$, then $N(\mathbf{v}) = \mathbf{v}$.

Theorem 3. Let $\mathbf{w} \in \mathbb{F}_q^n$. If $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) \leq 1$ then $\mathbf{w} \in \text{List}$.

Proof: We will proceed by induction on \mathbb{F}_q^n with the order \prec . The statement is true for $\mathbf{0} \in \mathbb{F}_q^n$. Now for the inductive step, we assume that the desired property is true for any word $\mathbf{u} \in \mathbb{F}_q^n$ such that $d_H(\mathbf{u}, \text{CL}(\mathcal{C})) \leq 1$ and also \mathbf{u} is smaller than an arbitrary but fixed $\mathbf{w} \setminus \{\mathbf{0}\}$ with respect to \prec and $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) \leq 1$, i.e.

if $d_H(\mathbf{u}, \text{CL}(\mathcal{C})) \leq 1$ and $\mathbf{u} \prec \mathbf{w}$ then $\mathbf{u} \in \text{List}$.

We will show that the previous conditions imply that \mathbf{w} is also in List .

Let $\mathbf{w} = \mathbf{v} + \mathbf{e}_{ij}$, with $(i, j) \in \text{supp}_{\text{gen}}(\mathbf{w})$ then $\mathbf{v} \prec \mathbf{w}$ by Eq. (2). If $d_H(\mathbf{v}, \text{CL}(\mathcal{C})) \leq 1$ then by the induction hypothesis we have that $\mathbf{v} \in \text{List}$ and by Criteria 1 or 2 in Definition 4 it is guaranteed that $\mathbf{w} \in \text{List}$. Then, let us suppose that $w_H(\mathbf{v}) = w_H(\text{CL}(\mathbf{v})) + 2$. Since $d_H(\mathbf{w}, \text{CL}(\mathcal{C})) \leq 1$ we have $\text{supp}_{\text{gen}}(\mathbf{v})[i] \neq \emptyset$ and $\text{supp}_{\text{gen}}(\mathbf{v}')[i] = \emptyset$ for all $\mathbf{v}' \in \text{CL}(\mathbf{v})$. Thus we can write \mathbf{v} as

$$\mathbf{v} = \mathbf{v}_0 + \sum_{j_k \in \text{supp}_{\text{gen}}(\mathbf{v})[i]} \mathbf{e}_{ij_k}, \text{ and } \text{supp}_{\text{gen}}(\mathbf{v}_0)[i] = \emptyset. \quad (3)$$

Let $L = |\text{supp}_{\text{gen}}(\mathbf{v})[i]|$ and let us renumber the partial sums of the words in the previous equation as $\mathbf{v}_l = \mathbf{v}_0 + \sum_{k=1}^l \mathbf{e}_{ij_k(l)}$, where $l = 1, \dots, L$. Note that $\mathbf{v}_L = \mathbf{v}$. It can be proved that

- 1) either there exists an element $k \in \{1, \dots, L-1\}$ such that $d_H(\mathbf{v}_k, \text{CL}(\mathcal{C})) = 1$
- 2) or for all $h \in \{1, \dots, L\}$ the word \mathbf{v}_h satisfies the same conditions as \mathbf{v} , that is $w_H(\mathbf{v}_h) = w_H(\text{CL}(\mathbf{v}_h)) + 2$ and $\text{supp}_{\text{gen}}(\mathbf{v}_h)[i] \neq \emptyset$ and $\text{supp}_{\text{gen}}(\mathbf{v}'_h)[i] = \emptyset$, for all $\mathbf{v}'_h \in \text{CL}(\mathbf{v}_h)$.

In the second case it can be proved that $d_H(\mathbf{v}_0, \text{CL}(\mathbf{v}_0)) = 1$; therefore in both cases there exists an element $k \in \{0, \dots, L-1\}$ such that $d_H(\mathbf{v}_k, \text{CL}(\mathbf{v}_k)) = 1$. Thus since $\mathbf{v}_k \prec \mathbf{w}$ and taking into account the induction hypothesis we have that $\mathbf{v}_k \in \text{List}$. By Criterion 2 in Definition 4, $\mathbf{v}_{k+1} \in \text{List}$ and hence with a successive application of Criterion 3 we have that $\mathbf{w} \in \text{List}$. ■

B. Leader codewords

Definition 5. The set of *leader codewords* of a linear code \mathcal{C} is defined as

$$\text{L}(\mathcal{C}) = \left\{ \begin{array}{l} \mathbf{n}_1 + \mathbf{e}_{ij} - \mathbf{n}_2 \in \mathcal{C} \setminus \{\mathbf{0}\} \mid \mathbf{n}_1 + \mathbf{e}_{ij} \in \text{SF}(\mathbb{F}_q^n), \\ \mathbf{n}_2 \in \text{CL}(\mathcal{C}), d_H(\mathbf{n}_1, \text{CL}(\mathcal{C})) \leq 1 \\ \text{and } d_H(\mathbf{n}_1 + \mathbf{e}_{ij}, \text{CL}(\mathcal{C})) \leq 1 \end{array} \right\}.$$

Note that the definition is a bit more complex than the one for binary codes in [3] due to the fact that in the general case not all coset leaders need to be ancestors of coset leaders. The name of leader codewords comes from the fact that one could compute all coset leaders of a corresponding word knowing the set $\text{L}(\mathcal{C})$ (see [3, Algorithm 3] that can be translated straightforward to the q -ary case).

Theorem 4 (Properties of $\text{L}(\mathcal{C})$). Let \mathcal{C} be a linear code then

- 1) $\text{L}(\mathcal{C})$ is a test set for \mathcal{C} .
- 2) Let \mathbf{w} be an element in $\text{L}(\mathcal{C})$ then

$$w_H(\mathbf{w}) \leq 2\rho(\mathcal{C}) + 1$$

where $\rho(\mathcal{C})$ is the covering radius of the code \mathcal{C} .

- 3) If $\mathbf{w} \in \text{L}(\mathcal{C})$ then

$$\mathcal{X}(\text{D}(\mathbf{0})) \cap (\text{D}(\mathbf{w}) \cup \mathcal{X}(\text{D}(\mathbf{w}))) \neq \emptyset.$$

- 4) If $\mathcal{X}(\text{D}(\mathbf{0})) \cap \text{D}(\mathbf{w}) \neq \emptyset$ then $\mathbf{w} \in \text{L}(\mathcal{C})$.

Proof: Items 1) and 2) follow directly from the definition of leader codewords and the proof of the same results in the binary case (see [3]).

3) Let $\mathbf{w} \in \text{L}(\mathcal{C})$, then $\mathbf{w} = \mathbf{n}_1 + \mathbf{e}_{ij} - \mathbf{n}_2$, where $\mathbf{n}_1, \mathbf{n}_2$ are elements in \mathbb{F}_q^n such that $\mathbf{n}_1 + \mathbf{e}_{ij} \in \text{SF}(\mathbb{F}_q^n)$, $\mathbf{n}_2 \in \text{CL}(\mathcal{C})$, $d_H(\mathbf{n}_1, \text{CL}(\mathcal{C})) \leq 1$ and $d_H(\mathbf{n}_1 + \mathbf{e}_{ij}, \text{CL}(\mathcal{C})) \leq 1$.

- If $\mathbf{n}_1 + \mathbf{e}_{ij} \notin \text{CL}(\mathcal{C})$, then $\mathbf{n}_1 + \mathbf{e}_{ij} \in \mathcal{X}(\text{D}(\mathbf{0}))$ and $(\mathbf{n}_1 + \mathbf{e}_{ij}) - \mathbf{w} = \mathbf{n}_2 \in \text{CL}(\mathcal{C})$ implies that $\mathbf{n}_1 + \mathbf{e}_{ij} \in \text{D}(\mathbf{w})$.
- If $\mathbf{n}_1 + \mathbf{e}_{ij} \in \text{CL}(\mathcal{C})$ we define $\mathbf{n}'_1 = \mathbf{n}'_{10} = \mathbf{n}_1 + \mathbf{e}_{ij}$. It is clear that $\mathbf{n}'_1, \mathbf{n}_2 \in \text{CL}(\mathbf{n}_2)$. Since $\mathbf{w} \neq \mathbf{0}$ let l be a number in the set $\{1, \dots, n\}$ such that $\mathbf{n}'_1[l] - \mathbf{n}_2[l] \neq 0$. Let $\mathbf{n}_2[l] = \sum_{j=1}^T \mathbf{e}_{li_j}$, for $1 \leq h \leq T$, $\mathbf{n}'_{1h} = \mathbf{n}'_1 + \sum_{j=1}^h \mathbf{e}_{li_j}$ and $\mathbf{n}_{2h} = \mathbf{n}_2 - \sum_{j=1}^h \mathbf{e}_{li_j}$. If there exists an h ($1 \leq h < T$) such that $\mathbf{n}'_{1h} \notin \text{CL}(\mathcal{C})$ and $\mathbf{n}'_{1h-1} \in \text{CL}(\mathcal{C})$ then these two conditions imply that $\mathbf{n}'_{1h} \in \mathcal{X}(\text{D}(\mathbf{0}))$ and that $\mathbf{n}_{2h} = \mathbf{n}'_{1h} - \mathbf{w}$ is either a coset leader ($\mathbf{n}'_{1h} \in \text{D}(\mathbf{w})$) or $d_H(\mathbf{n}_{2h}, \mathbf{n}_2) = 1$ ($\mathbf{n}'_{1h} \in \mathcal{X}(\text{D}(\mathbf{w}))$).

If there is no such h ($1 \leq h < T$) satisfying the condition then $w_H(\mathbf{n}'_{1T}) = w_H(\mathbf{n}_{2T}) + 1$, which means that \mathbf{n}'_{1T} is not a coset leader and \mathbf{n}'_{1T-1} is a coset leader. Then using the same idea of the previous paragraph we have that $\mathbf{n}'_{1T} \in \mathcal{X}(\text{D}(\mathbf{0}))$ and $\mathbf{n}'_{1T} \in \text{D}(\mathbf{w}) \cup \mathcal{X}(\text{D}(\mathbf{w}))$.

4) If $\mathcal{X}(\text{D}(\mathbf{0})) \cap \text{D}(\mathbf{w}) \neq \emptyset$, let $\mathbf{v} \in \mathcal{X}(\text{D}(\mathbf{0})) \cap \text{D}(\mathbf{w})$. The first condition $\mathbf{v} \in \mathcal{X}(\text{D}(\mathbf{0}))$ implies $\mathbf{v} = \mathbf{n}_1 + \mathbf{e}_{ij}$ for some \mathbf{n}_1 such that $d_H(\mathbf{n}_1, \text{CL}(\mathcal{C})) \leq 1$ and $(i, j) \in \text{supp}_{\text{gen}}(\mathbf{v})$. On the other hand, $\mathbf{n}_1 + \mathbf{e}_{ij} \in \text{D}(\mathbf{w})$ implies that $\mathbf{n}_2 = (\mathbf{n}_1 + \mathbf{e}_{ij}) - \mathbf{w} \in \text{CL}(\mathcal{C})$. Therefore, $\mathbf{w} = \mathbf{n}_1 + \mathbf{e}_{ij} - \mathbf{n}_2 \in \text{L}(\mathcal{C})$. ■

Remark 3. Note that item 3) in Theorem 4 implies that any leader codeword is a zero neighbour however, this is one of the differences with the binary case, it is not always true that for a leader codewords \mathbf{w} we have that $\mathcal{X}(\text{D}(\mathbf{0})) \cap \text{D}(\mathbf{w}) \neq \emptyset$, although by item 4) we have that \mathbf{w} is a leader codeword provided this condition is satisfied. Furthermore, 4) guarantees that the set of leader codewords contains all the minimal test set according to its cardinality (see [1]). As a conclusion we could say because of all this properties in Theorem 4 that the

the set of leader codewords is a “good enough” subset of the set of zero neighbours.

IV. CORRECTABLE AND UNCORRECTABLE ERRORS

We define the relation \subset_1 in the additive monoid which describe exactly the relation \subset in the vector space \mathbb{F}_q^n . Given $\mathbf{x}, \mathbf{y} \in (\mathbb{F}_q^n, +)$

$$\mathbf{x} \subset_1 \mathbf{y} \text{ if } \mathbf{x} \subset \mathbf{y} \text{ and } \text{supp}_{\text{gen}}(\mathbf{x}) \cap \text{supp}_{\text{gen}}(\mathbf{y} - \mathbf{x}) = \emptyset. \quad (4)$$

Note that this definition translates the binary case situation in [4]. In this case given a $\mathbf{y} \in (\mathbb{F}_q^n, +)$ there are more words $\mathbf{x} \in (\mathbb{F}_q^n, +)$ such that $\mathbf{x} \subset \mathbf{y}$ than if we consider \mathbf{x}, \mathbf{y} as elements in the vector space \mathbb{F}_q^n . Of course, any relation $\mathbf{x} \subset \mathbf{y}$ in \mathbb{F}_q^n as a vector space is also true in the additive monoid, but it is not true the other way round.

The set $E^0(\mathcal{C})$ of *correctable errors* of a linear code \mathcal{C} is the set of the minimal elements with respect to \prec in each coset. The elements of the set $E^1(\mathcal{C}) = \mathbb{F}_q^n \setminus E^0(\mathcal{C})$ will be called *uncorrectable errors*. A *trial set* $T \subset \mathcal{C} \setminus \mathbf{0}$ of the code \mathcal{C} is a set which has the following property

$$\mathbf{y} \in E^0(\mathcal{C}) \text{ if and only if } \mathbf{y} \leq \mathbf{y} + \mathbf{c}, \text{ for all } \mathbf{c} \in T.$$

Since \prec is a monotone α -ordering on \mathbb{F}_q^n , the set of correctable and uncorrectable errors form a monotone structure. Namely, if $\mathbf{x} \subset_1 \mathbf{y}$ then $\mathbf{x} \in E^1(\mathcal{C})$ implies $\mathbf{y} \in E^1(\mathcal{C})$ and $\mathbf{y} \in E^0(\mathcal{C})$ implies $\mathbf{x} \in E^0(\mathcal{C})$. In the general case $q \neq 2$ there is a difference with respect to the binary case, there may be words $\mathbf{y}' \in \mathbb{F}_q^n$ s.t. $\text{supp}_{\text{gen}}(\mathbf{y}') = \text{supp}_{\text{gen}}(\mathbf{y})$, $\mathbf{y}' \subset \mathbf{y}$ and \mathbf{y}' could be either a correctable error or an uncorrectable error, so, the monotone structure it is not sustained by \subset in the additive monoid $(\mathbb{F}_q^n, +)$. Note that in Definition 4, the set denoted by \mathcal{N} is the set $E^0(\mathcal{C})$ corresponding to \prec .

Let the set of minimal uncorrectable errors $M^1(\mathcal{C})$ be the set of $\mathbf{y} \in E^1(\mathcal{C})$ such that, if $\mathbf{x} \subset_1 \mathbf{y}$ and $\mathbf{x} \in E^1(\mathcal{C})$, then $\mathbf{x} = \mathbf{y}$. In a similar way, the set of maximal correctable errors is the set $M^0(\mathcal{C})$ of elements $\mathbf{x} \in E^0(\mathcal{C})$ such that, if $\mathbf{x} \subset_1 \mathbf{y}$ and $\mathbf{y} \in E^0(\mathcal{C})$, then $\mathbf{x} = \mathbf{y}$.

For $\mathbf{c} \in \mathcal{C} \setminus \mathbf{0}$, a *larger half* is defined as a minimal word \mathbf{u} in the ordering \preceq such that $\mathbf{u} - \mathbf{c} \prec \mathbf{u}$. The weight of such a word \mathbf{u} is such that

$$w_H(\mathbf{c}) \leq 2w_H(\mathbf{u}) \leq w_H(\mathbf{c}) + 2,$$

see [4] for more details. The set of larger halves for a codeword \mathbf{c} is denoted by $L(\mathbf{c})$, and for $U \subseteq \mathcal{C} \setminus \mathbf{0}$ the set of larger halves for elements of U is denoted by $L(U)$. Note that $L(\mathcal{C}) \subseteq E^1(\mathcal{C})$.

For any $\mathbf{y} \in \mathbb{F}_q^n$, let $H(\mathbf{y}) = \{\mathbf{c} \in \mathcal{C} : \mathbf{y} - \mathbf{c} \prec \mathbf{y}\}$, and we have $\mathbf{y} \in E^0(\mathcal{C})$ if and only if $H(\mathbf{y}) = \emptyset$, and $\mathbf{y} \in E^1(\mathcal{C})$ if and only if $H(\mathbf{y}) \neq \emptyset$.

In [4, Theorem 1] there is a characterization of the set $M^1(\mathcal{C})$ in terms of $H(\cdot)$ and larger halves of the set of minimal codewords $M(\mathcal{C})$ for the binary case. It is easy to proof that this Theorem and [4, Corollary 3] are also true for any linear code.

Proposition 1 (Corollary 3 in [4]). *Let \mathcal{C} be a linear code and $T \subseteq \mathcal{C} \setminus \mathbf{0}$. The following statements are equivalent:*

- 1) T is a trial set for \mathcal{C} .

- 2) If $\mathbf{y} \in M^1(\mathcal{C})$, then $T \cap H(\mathbf{y}) \neq \emptyset$.
- 3) $M^1(\mathcal{C}) \subseteq L(T)$.

Now we will formulate the result which relates the trial sets for a given weight compatible order \prec and the set of leader codewords.

Theorem 5. *Let \mathcal{C} be a linear code and $L(\mathcal{C})$ the set of leaders codewords for \mathcal{C} , the following statements are satisfied.*

- 1) $L(\mathcal{C})$ is a trial set for any given \prec .
- 2) $L(\mathcal{C})$ contains any minimal trial set for any given \prec .
- 3) Algorithm 2 in [3] can be adapted to compute a set of leader codewords which is a trial set T for a given \prec such that it satisfies the following property

$$\text{For any } \mathbf{c} \in T, \text{ there exists } \mathbf{y} \in M^1(\mathcal{C}) \cap L(\mathcal{C}) \text{ s.t. } \mathbf{y} - \mathbf{c} \in E^0(\mathcal{C}).$$

Proof:

Proof of 1) We will prove statement 2 of Proposition 1. Let $\mathbf{y} \in M^1(\mathcal{C})$, let i such that $\text{supp}_{\text{gen}}(\mathbf{y})[i] \neq \emptyset$ and $\mathbf{n}_1 = \mathbf{y} - \mathbf{y}_i$. Since $\mathbf{y} \in M^1(\mathcal{C})$ we have that $\mathbf{n}_1 \in E^0(\mathcal{C})$, thus it is also a coset leader. On the other hand, let $\mathbf{n}_2 \in E^0(\mathcal{C})$ such that $\mathbf{n}_2 \in \text{CL}(\mathbf{y})$ and $\mathbf{c} = \mathbf{y} - \mathbf{n}_2$. It is clear that \mathbf{c} is a leader codeword and $\mathbf{y} - \mathbf{c} = \mathbf{n}_2 \prec \mathbf{y}$. Therefore $\mathbf{c} \in H(\mathbf{y})$.

Proof of 2) By statement 3 of Proposition 1, a minimal trial set should contain at least as larger halves the elements in the set $M^1(\mathcal{C})$. But it can be seen that a set of codewords having as larger halves at least the set $M^1(\mathcal{C})$ is already a trial set.

Let us show that any codeword with a larger half belonging to $M^1(\mathcal{C})$ is a leader codeword. Let $\mathbf{c} \in \mathcal{C}$ such that $L(\mathbf{c}) \cap M^1(\mathcal{C}) \neq \emptyset$. Then there exists an element $\mathbf{y} \in M^1(\mathcal{C})$ such that $\mathbf{y} \in L(\mathbf{c})$. Thus $\mathbf{c} = \mathbf{y} - \mathbf{n}_2$ with $\mathbf{n}_2 \prec \mathbf{y}$. On the other hand let $i \in \{1, \dots, n\}$ such that $\text{supp}_{\text{gen}}(\mathbf{y})[i] \neq \emptyset$, and let $\mathbf{n}_1 = \mathbf{y} - \mathbf{y}_i$. Then $\mathbf{y} \in M^1(\mathcal{C})$ implies that $\mathbf{n}_1 \in E^0(\mathcal{C})$ and therefore it is a coset leader. from the fact $\mathbf{n}_2 \prec \mathbf{y}$ we have two cases,

- i.- $w_H(\mathbf{y}) > w_H(\mathbf{n}_2)$, in this case since $\mathbf{n}_1 \in \text{CL}(\mathcal{C})$ we have that $\mathbf{n}_2 \in \text{CL}(\mathcal{C})$ and then \mathbf{c} is a leader codeword.
- ii.- $w_H(\mathbf{y}) = w_H(\mathbf{n}_2)$, then $d_H(-\mathbf{n}_2 + \mathbf{y}_i, \text{CL}(\mathcal{C})) \leq 1$, then, $\mathbf{c} = -\mathbf{n}_2 + \mathbf{y}_i + \mathbf{n}_1$ is a leader codeword.

Proof of 3) In the steps of the construction of the leader codewords in Algorithm 2 (Steps 11 - 13 for the binary case) it is enough to state the condition $\mathbf{t} \in M^1(\mathcal{C})$ for \mathbf{t} and take \mathbf{t}_k only equal to the coset leader of $\text{CL}(\mathbf{t})$, that is the corresponding correctable error and add the codeword $\mathbf{t} - \mathbf{t}_k$ to the set $L(\mathcal{C})$. ■

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